

Dissipative Heating in a Rotational Viscometer with Coaxial Cylinders

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Rotational viscometers with coaxial cylinders are often used for measuring rheological behaviour. If the inner to outer cylinder diameter ratio does not differ significantly from 1, the curvature can be neglected and the flow reduces to the flow between moving and stationary plates. The power-law and Bingham models are often used for describing rheological behaviour. This paper deals with the temperature distribution obtained by solving the Fourier-Kirchhoff equation and in the case of negligible inner heat resistance it also covers temperature time dependence. The solution is illustrated by a numerical example.

Keywords: viscous heating, viscometer with coaxial cylinders, power-law fluids, Bingham plastics.

1 Introduction

This paper presents viscous heating in a rotational viscometer with coaxial cylinders (see Fig. 1), which is often used for measuring rheological behaviour.

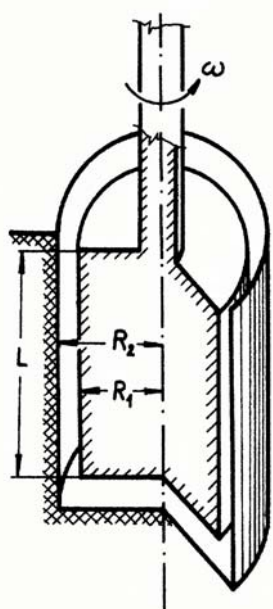


Fig. 1: Rotational viscometer with coaxial cylinders

Power-law and Bingham models are often used to describe rheological behaviour [1].

The power-law model is the simplest model widely used for describing the rheological behaviour of non-Newtonian fluids. Using this model, the dependence of shear stress τ on shear rate can be expressed by the relation:

$$\tau = K\dot{\gamma}^n, \quad (1)$$

where K is the coefficient of consistency and n stands for the flow behaviour index.

The Bingham model is the simplest model used for describing the rheological behaviour of viscoplastic materials. Using this model, the relation of shear stress τ and shear rate $\dot{\gamma}$ can be expressed by the following relation:

$$\tau = \mu_p \dot{\gamma} + \tau_0 \text{ for } |\tau| \geq \tau_0, \quad (2)$$

where τ_0 is the yield stress and μ_p stands for plastic viscosity.

If the inner to outer cylinder diameter ratio does not differ significantly from 1, the curvature can be neglected and the flow reduces to the flow between the moving and stationary plates.

The temperature distribution can be obtained by solving the Fourier-Kirchhoff equation [1]

$$\lambda \frac{d^2 T}{dy^2} = -\tau \dot{\gamma}, \quad (3)$$

where y is the distance from the stationary plate. The equation will be solved with the following boundary conditions

$$\begin{aligned} y = H, \quad \frac{dT}{dy} &= 0 \\ y = 0, \quad \lambda \frac{dT}{dy} &= \alpha(T - T_f) \end{aligned} \quad (4)$$

i.e., we assume an insulated moving plate (rotating cylinder) and a stationary plate (cylinder) tempered to temperature T_f .

2 Solution

2.1 Power-law fluids

Inserting (1) for τ into (3), we get

$$\lambda \frac{d^2 T}{dy^2} = -K \dot{\gamma}^{n+1} \quad (5)$$

and after integration we obtain

$$T - T_f = \frac{KH^2 \dot{\gamma}^{n+1}}{\lambda} \left(\frac{y}{H} - \frac{1}{2} \left(\frac{y}{H} \right)^2 + \frac{1}{Bi} \right), \quad (6)$$

where $Bi = \alpha H / \lambda$. The solution is shown in graphical form in Fig. 2, where

$$T^* = \frac{(T - T_f)\lambda}{KH^2 \dot{\gamma}^{n+1}}, \quad y^* = \frac{y}{H}. \quad (7)$$

2.2 Bingham plastics

Inserting (2) for τ into (3), we get

$$\lambda \frac{d^2 T}{dy^2} = -\frac{1}{\lambda} (\mu_p \dot{\gamma}^2 + \tau_0 \dot{\gamma}) \quad (8)$$

after integration and rearrangement we obtain

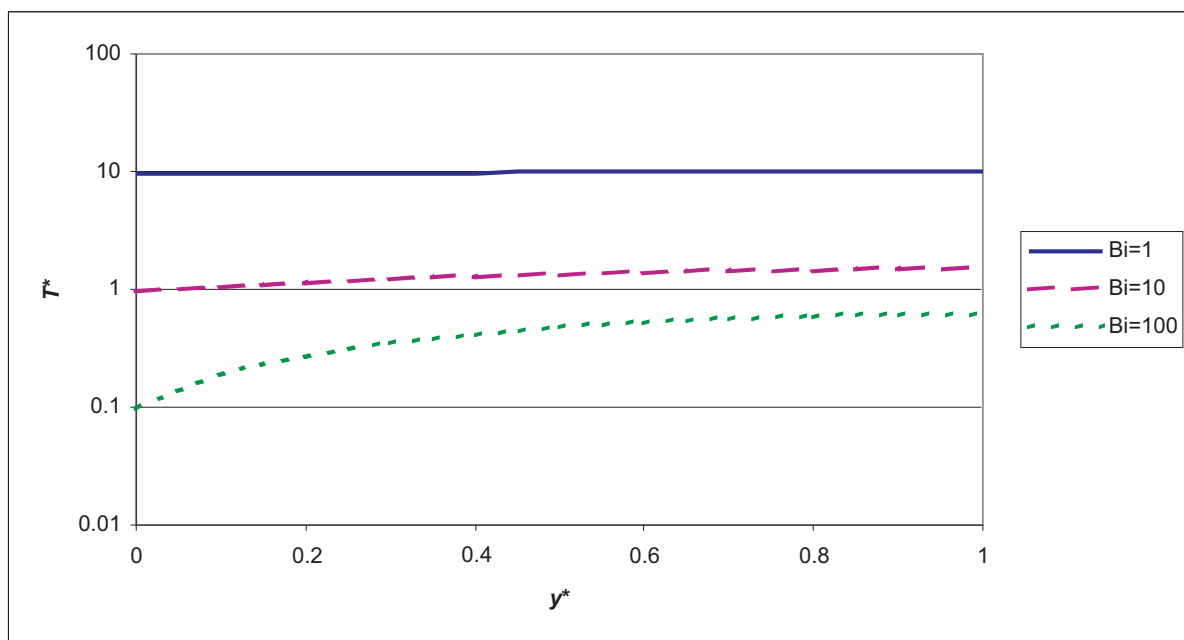


Fig. 2: Dimensionless temperature profiles

$$T - T_f = \frac{\mu_p H^2 \dot{\gamma}^2}{\lambda} (1 + \tau_0^*) \left(\frac{y}{H} - \frac{1}{2} \left(\frac{y}{H} \right)^2 + \frac{1}{Bi} \right), \quad (9)$$

where

$$\tau_0^* = \frac{\tau_0}{\mu_p \dot{\gamma}}. \quad (10)$$

The solution of (9) for $Bi \rightarrow \infty$ is shown in graphical form in Fig. 3, where

$$T^* = \frac{(T - T_f) \lambda}{\mu_p H^2 \dot{\gamma}^2}. \quad (11)$$

Fig. 3 shows that increasing plasticity (τ_0^*) increases the temperature rise.

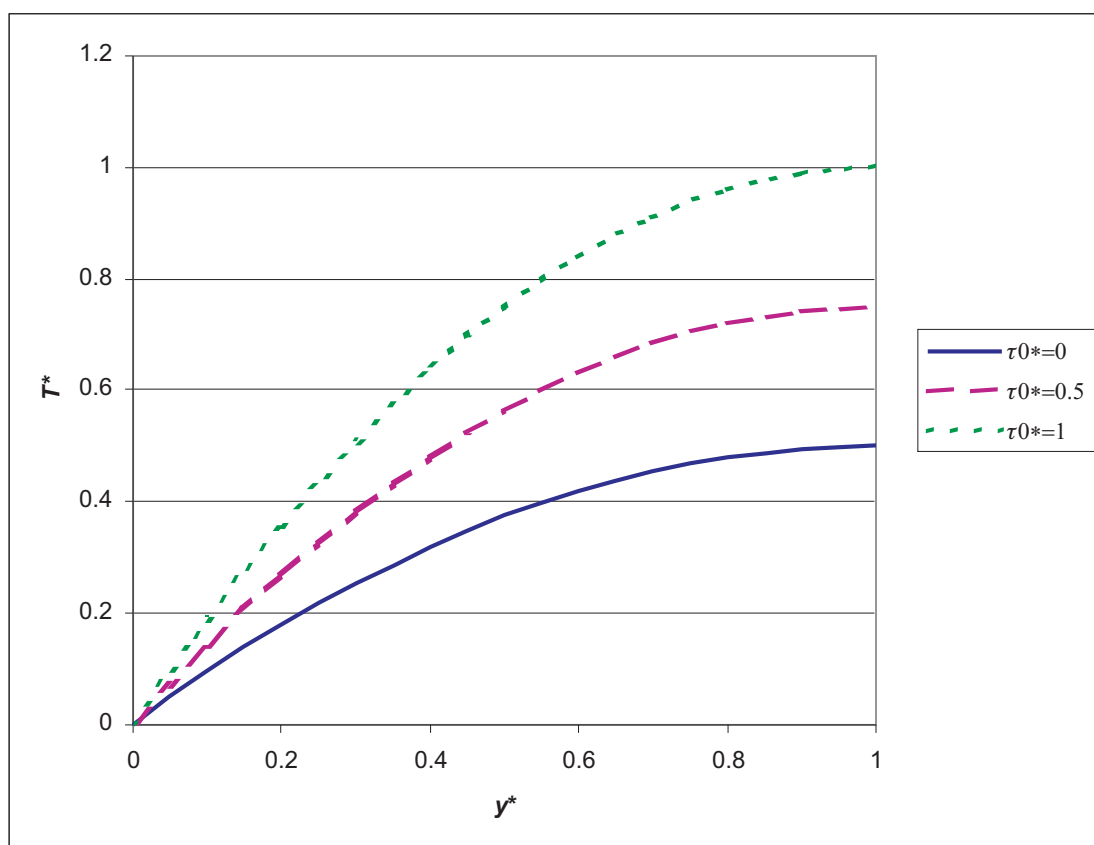


Fig. 3: Dimensionless temperature profiles

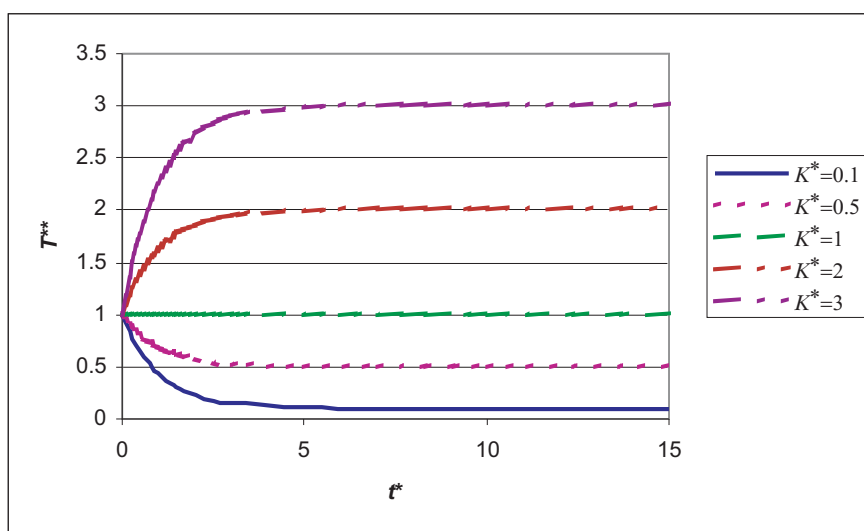


Fig. 4: Dependence of dimensionless temperature on dimensionless time

From the dependencies shown in Fig. 2 it can be seen that at small Biot number values the outer temperature resistance prevails and the temperature of the liquid is practically constant, and in a unsteady-state it depends on time. For this case the enthalpy balance can be written in the form

$$\rho c H \frac{dT}{dt} = \tau \dot{\gamma} H - \alpha(T - T_f) \quad (12)$$

and after integration it transforms to

$$T^{**} = \frac{T - T_f}{T_0 - T_f} = K^* + (1 - K^*) \exp(-t^*), \quad (13)$$

where

$$K^* = \frac{\tau \dot{\gamma} H}{\alpha(T_0 - T_f)}, \quad t^* = \frac{\alpha t}{\rho c H}. \quad (14)$$

The dependence is shown in Fig. 4, where the line for $K^* = 1$ designates the equilibrium state at which all dissipative heat is removed by convection.

The dependence of the dimensionless time t^* after which the dimensionless temperature attains 99 % of the steady-state value on K^* is shown in Fig. 5.

In the case when initial temperature T_0 is equal to temperature T_f it is suitable to define the dimensionless temperature as

$$T^+ = \frac{(T - T_f)\alpha}{\tau \dot{\gamma} H} \quad (15)$$

after integration (11) we can obtain the relation

$$T^+ = 1 - \exp(-t^*). \quad (16)$$

The application of the above relations will be illustrated in the following example.

3 Example

A rotational viscometer with inner cylinder diameter 48 mm and outer cylinder diameter 50 mm contains a Newtonian liquid with density $\rho = 1000 \text{ kg m}^{-3}$, heat capacity $c = 4200 \text{ J kg}^{-1} \text{ K}^{-1}$ and heat conductivity $\lambda = 0.5 \text{ W m}^{-1} \text{ K}^{-1}$. Calculate the inner and outer cylinder temperature

- 1) when the tempering temperature is 20°C and the heat transfer coefficient $\alpha = 100 \text{ W m}^{-2} \text{ K}^{-1}$

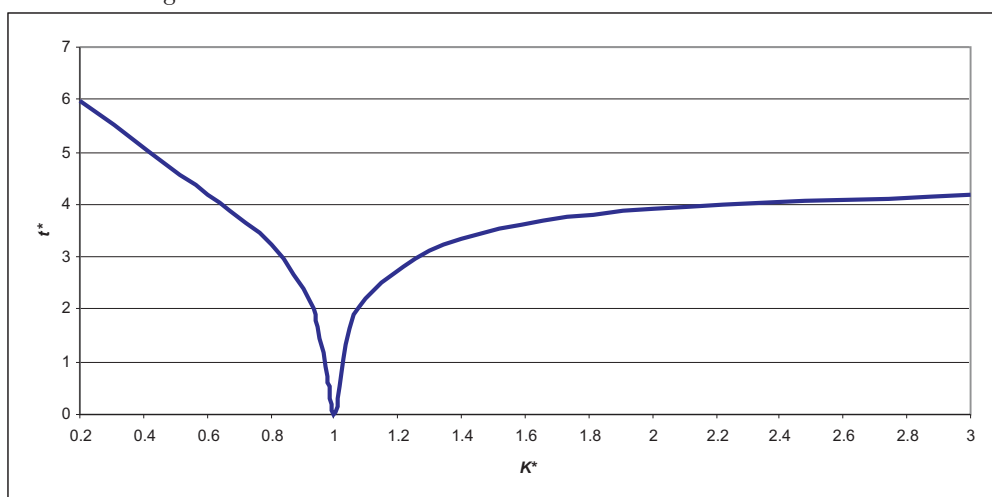


Fig. 5: Dependence of the dimensionless time t^* after which the dimensionless temperature attains 99 % of the steady-state value on K^*

- 2) at no tempering, outer temperature 20 °C and $\alpha = 5 \text{ Wm}^{-2}\text{K}^{-1}$. The temperature dependence of viscosity is described by the relation

$$\mu [\text{Pa}\cdot\text{s}] = 0.82 \exp(-0.025 T [^\circ\text{C}]).$$

Measurement is carried out at shear rate

- a) 100 s^{-1} ,
b) 1000 s^{-1} .

Calculate also the temperature after 5 minutes, when the initial temperature is

- α) 20 °C,
 β) 15 °C.

4 Solution

4.1 Calculations of final temperatures

First the Biot number will be calculated

$$1) Bi = \frac{\alpha H}{\lambda} = \frac{100 \times 0.001}{0.5} = 0.2.$$

Inserting $n = 1$ and $K = \mu$ into Eq. (6) or inserting $\tau_0^* = 0$ and $\mu_p = \mu$ into Eq. (9), the inner cylinder temperature T_i will be calculated for $y = H$ from

$$\begin{aligned} a) T_i &= \frac{\mu H^2 \dot{\gamma}^2}{\lambda} \left(\frac{1}{2} + \frac{1}{Bi} \right) + T_f \\ &= \frac{0.5 \times 10^{-6} \times 10^4}{0.5} 5.5 + 20 = 20.055 \text{ } ^\circ\text{C} \end{aligned}$$

and the outer cylinder temperature T_e will be calculated for $y = 0$ from

$$\begin{aligned} T_e &= \frac{\mu H^2 \dot{\gamma}^2}{\lambda} \left(0 + \frac{1}{Bi} \right) + T_f \\ &= \frac{0.5 \times 10^{-6} \times 10^4}{0.5} 5 + 20 = 20.05 \text{ } ^\circ\text{C}. \end{aligned}$$

$$\begin{aligned} b) T_i &= \frac{\mu H^2 \dot{\gamma}^2}{\lambda} \left(\frac{1}{2} + \frac{1}{Bi} \right) + T_f \\ &= \frac{0.44 \times 10^{-6} \times 10^6}{0.5} 5.5 + 20 = 24.9 \text{ } ^\circ\text{C}, \end{aligned}$$

$$\begin{aligned} T_e &= \frac{\mu H^2 \dot{\gamma}^2}{\lambda} \frac{1}{Bi} + T_f \\ &= \frac{0.44 \times 10^{-6} \times 10^6}{0.5} 5 + 20 = 24.4 \text{ } ^\circ\text{C}. \end{aligned}$$

The viscosities at the mean liquid temperature were inserted into the above equations.

$$2) Bi = \frac{\alpha H}{\lambda} = \frac{5 \times 0.001}{0.5} = 0.01$$

and the two temperatures will be calculated from the same equations

$$\begin{aligned} a) T_i &= \frac{\mu H^2 \dot{\gamma}^2}{\lambda} \left(\frac{1}{2} + \frac{1}{Bi} \right) + T_f \\ &= \frac{0.485 \times 10^{-6} \times 10^4}{0.5} 100.5 + 20 = 21 \text{ } ^\circ\text{C}, \end{aligned}$$

$$\begin{aligned} T_e &= \frac{\mu H^2 \dot{\gamma}^2}{\lambda} \left(0 + \frac{1}{Bi} \right) + T_f \\ &= \frac{0.485 \times 10^{-6} \times 10^4}{0.5} 100 + 20 = 21 \text{ } ^\circ\text{C}. \end{aligned}$$

b)

$$\begin{aligned} T_i &= \frac{\mu H^2 \dot{\gamma}^2}{\lambda} \left(\frac{1}{2} + \frac{1}{Bi} \right) + T_f \\ &= \frac{0.19 \times 10^{-6} \times 10^6}{0.5} 100.5 + 20 = 58.3 \text{ } ^\circ\text{C}, \end{aligned}$$

$$\begin{aligned} T_e &= \frac{\mu H^2 \dot{\gamma}^2}{\lambda} \frac{1}{Bi} + T_f \\ &= \frac{0.19 \times 10^{-6} \times 10^6}{0.5} 100 + 20 = 58.2 \text{ } ^\circ\text{C}. \end{aligned}$$

These results show that at a high shear rate the temperature rise is unacceptable especially without tempering. It can also be seen that the difference between the inner and outer cylinder temperature is not high due to the low Biot number values, especially in case 2).

4.2 Calculations of temperatures after 5 minutes of measurement

- a) In the case when the initial temperature T_0 is equal to temperature T_f , Eq.(15) will be used in the calculations

$$\begin{aligned} 1a) T &= \frac{\mu H \dot{\gamma}^2}{\alpha} [1 - \exp(-t^*)] + T_f \\ &= \frac{0.497 \times 0.001 \times 100^2}{100} \left[1 - \exp \frac{-100 \times 300}{1000 \times 4200 \times 0.001} \right] + 20 \\ &= 20.05 \text{ } ^\circ\text{C}, \end{aligned}$$

$$\begin{aligned} 1b) T &= \frac{\mu H \dot{\gamma}^2}{\alpha} [1 - \exp(-t^*)] + T_f \\ &= \frac{0.47 \times 0.001 \times 1000^2}{100} \left[1 - \exp \frac{-100 \times 300}{1000 \times 4200 \times 0.001} \right] + 20 \\ &= 24.7 \text{ } ^\circ\text{C}, \end{aligned}$$

$$\begin{aligned} 2a) T &= \frac{\mu H \dot{\gamma}^2}{\alpha} [1 - \exp(-t^*)] + T_f \\ &= \frac{0.5 \times 0.001 \times 100^2}{5} \left[1 - \exp \frac{-5 \times 300}{1000 \times 4200 \times 0.001} \right] + 20 \\ &= 20.3 \text{ } ^\circ\text{C}, \end{aligned}$$

$$\begin{aligned} 2b) T &= \frac{\mu H \dot{\gamma}^2}{\alpha} [1 - \exp(-t^*)] + T_f \\ &= \frac{0.375 \times 0.001 \times 1000^2}{5} \left[1 - \exp \frac{-5 \times 300}{1000 \times 4200 \times 0.001} \right] + 20 \\ &= 42.5 \text{ } ^\circ\text{C}, \end{aligned}$$

where viscosity was calculated at the mean temperature $(T + T_f)/2$.

β) In cases when the initial temperature T_0 is not equal to temperature T_f , Eq.(12) will be used in the calculations

$$\begin{aligned} 1a) \quad T &= T_f + (T_0 - T_f)[K^* + (1 - K^*) \exp(-t^*)] \\ &= 20 - 5[(1 + 0.0106) \exp(-t^*) - 0.0106] = 20.05 \text{ }^\circ\text{C}, \end{aligned}$$

$$\begin{aligned} 1b) \quad T &= T_f + (T_0 - T_f)[K^* + (1 - K^*) \exp(-t^*)] \\ &= 20 - 5[(1 + 0.995) \exp(-t^*) - 0.995] = 25 \text{ }^\circ\text{C}, \end{aligned}$$

$$\begin{aligned} 2a) \quad T &= T_f + (T_0 - T_f)[K^* + (1 - K^*) \exp(-t^*)] \\ &= 20 - 5[(1 + 0.22) \exp(-t^*) - 0.22] = 16.8 \text{ }^\circ\text{C}, \end{aligned}$$

$$\begin{aligned} 2b) \quad T &= T_f + (T_0 - T_f)[K^* + (1 - K^*) \exp(-t^*)] \\ &= 20 - 5[(1 + 16.3) \exp(-t^*) - 16.3] = 41 \text{ }^\circ\text{C}, \end{aligned}$$

where viscosity was calculated at the mean temperature $(T_0 + T)/2$.

The results presented above show that the temperature rise and the experimental error is considerable especially at high shear rate in cases 1b) and 2b).

5 Conclusion

It was shown that dissipative heating can play an important role in measurement of highly viscous fluids. The temperature of measured liquid can be significantly higher than tempering temperature, which can cause significant experimental error. The time necessary for temperature stabilisation is often not negligible. Measurement without tempering can lead to a significant temperature rise and unacceptable error of measurement.

List of symbols

Bi	Biot number, 1
c	specific heat capacity, $\text{J}\cdot\text{kg}^{-1}\cdot\text{K}^{-1}$
H	distance of planes (cylinder walls), m
K	consistency coefficient, $\text{Pa}\cdot\text{s}^n$
n	flow index, 1

R_1	inner cylinder radius, m
R_2	outer cylinder radius, m
t	time, s
T	temperature, K
y	coordinate, m
α	heat transfer coefficient, $\text{W}\cdot\text{m}^{-2}\cdot\text{K}^{-1}$
$\dot{\gamma}$	shear rate, s^{-1}
κ	ratio R_1/R_2 , 1
λ	heat conductivity, $\text{W}\cdot\text{m}^{-1}\cdot\text{K}^{-1}$
μ	viscosity, $\text{Pa}\cdot\text{s}$
μ_p	plastic viscosity, $\text{Pa}\cdot\text{s}$
ρ	density, $\text{kg}\cdot\text{m}^{-3}$
τ	shear stress, Pa
τ_0	yield stress, Pa
lower indexes	
f	fluid
0	initial
upper indices	
*+	dimensionless

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Reference

- [1] Bird, R. B., et al.: *Transport Phenomena*, J. Wiley, N. York, 1960.

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